# III. *Specimen Methodi Generalis determinandi Figurarum Quadraturas. Autore* Jo. Craig.

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## *Ad D. Georgium Cheynæum , M. D.*

FAcile credas, Vir Eruditiſſime, mihi non parum arridere, quòd Methodus, quâ uſus ſum in determinandis Figurarum Quadraturis, tantopere à D. Leibnitio & te probata fuerit; ut ille alteri cuidam à ſe inventæ non-nihil ſimilem agnoſceret. Tu verò ut conjecturam feceris ei non multo abſimilem eſſe illam quâ utitur D. Newtonus; eundemque ipſe tanto cùm ſucceſſu ſequaris, ut Methodus Calculi differentialis inverſa incredibili incremento jam à te promota ſit in Libro tuo, quem D. Archibaldo Pitcarnio Patriæ noſtræ & ſœculi, hujus Ornamento inſcripſiſti. Sed multa (ut nôſti) ad Methodum illam inverſam perficiendam neceſſaria adhuc invenienda ſuperſunt. Ego Tibi rerum harum judici peritiſſimo rationes aliquot breviter exponam, quæ faciunt ut reliqua per Methodus hactenus uſurpatas non poſſe obtineri putem.

Et primò quidem, cùm ex data relatione inter $z$ & $y$, quæritur $\int: z\,dy$, omnes illæ Methodi poſtulant ut $z$ per $y$ & datas exprimatur; quòd tamen fieri non poteſt, quando æquatio relationem illam definiens ultra Cubicam vel Bi-quadraticam aſcendit. Nam, non ſine magno hujus ſcientiæ opprobrio, hæret hîc adhuc Algebra Vulgaris. Secundo, quamvis regula innoteſceret generalis inveniendi Radices æquationum cujuſcunque gradûs; huic tamen Methodo inverſæ prorſus foret inutilis: Radix enim ſurdis tam complicatis involveretur, ut nullâ arte (hactenus cognitâ) à differentiali ad Integralem regreſſus dari poſſet. Ob has rationes, Vîr clariſſimè, aliàs vias & (favente Deo) conatu non prorſus irrito rem ſum aggreſſus. Et quia in his me quædam tibi non diſplicitura inveniſſe ſpero, ideo eorum Specimen Tibi publicè impertiri conſtitui. Precor interim ut Deus Otium tibi & vitam, ut Geometriam ac Medicinam ad talem perducas perfectionem, qualem in utrâque editæ à Te Primitiæ meritò expectare faciunt.

Amicum Tibi devinctiſſimum,

*Gillingham,*
22 *Apr.* 1703.

*Jo. Craig.*

## SECTIO I.

SIT $z^m + a y^n = b z^e y^r$ æquatio exprimens relationem inter Ordinatam $z$ & abciſſam $y$, in qua Exponentes $m, n, e, r$, denotant quoſlibet Numeros, Integros vel Fractos, Affirmativos vel Negativos. Ponatur $r - n = c$ Erit

$$\text{AREA} = \frac{m}{m+n} z y +$$

$$\frac{mc + ne}{m \times \overline{m+n} \times \overline{c+1} + n \times \overline{m+n} \times \overline{e+1}} + \frac{b}{a} z^{e+1} y^{c+1}$$

$$\frac{\overline{m-e}\times\overline{c+1}+r\times\overline{e+1}}{m\times\overline{2c+1}+n\times\overline{2c+1}}+\frac{bB}{a}z^{2c+1}y^{2c+1}+$$

$$\frac{\overline{m-e}\times\overline{2c+1}+r\times\overline{2c+1}}{m\times\overline{3c+1}+n\times\overline{3c+1}}+\frac{bC}{a}z^{3c+1}y^{3c+1}+$$

$$\frac{\overline{m-e}\times\overline{3c+1}+r\times\overline{3c+1}}{m\times\overline{4c+1}+n\times\overline{4c+1}}+\frac{bD}{a}z^{4c+1}y^{4c+1}+$$

$$\frac{\overline{m-e}\times\overline{4c+1}+r\times\overline{4c+1}}{m\times\overline{5c+1}+n\times\overline{5c+1}}+\frac{bE}{a}z^{5c+1}y^{5c+1}+\&c.$$

De hâc Seris hæc ſunt notanda : (1.) Quòd literæ majuſculæ B, C, D &c. deſignent coefficientes terminorum ipſis immediatè præcedentium : (2.) Quòd exhibeat Quadraturas omnium Figurarum Quadrabilium, quarum Curvæ per æquationem trium terminorum definiuntur : (3.) Et quòd ſemper ſint Quadrabiles, quando $\frac{m \times r - r}{mn - rn - en}$ eſt numerus integer & affirmativus, quem vocemus l. (4.) Speciatim l+1 dat numerum Terminorum (ab initio ſumptorum) Seriei Aream quæſitam conſtituentium : (5.) Quòd ſi ponatur e = 0, mutabitur hæc Series in Celebre Theorema Newtonianum pro Binomio communi; & proinde hoc Theorema eſt hujus Seriei caſus ſpecialis & ſimplex: (6.) Cùm ſit Applicatio hujus Seriei ad Figuram particularem, hæ regulæ ſunt obſervandæ. $1^a$ Reducatur æquatio Curvam datam definiens ad formam generalem; & ex comparatione particularis cùm generali invenientur coefficientes a, b; ut & exponentes m, n, e, r. Secunda, Si exponentes ſic determinati non faciant l numerum integrum & affirmativum, (juxta conditionem in Not. 3. aſſignatam,) tum alius terminus æquationis particularis à quantitate z liberetur; & ſi nunc exponentibus denuo determinatis non competat illa Quadrabilitatis conditio, tum reliquus terminus à quantitate z liberetur: Nam nullo labore quilibet ex tribus terminis æquationem datam conſtituentibus à quantitate z liberari poteſt. Tertia, Si æquationi per Regulam præcedentem tractatæ non conveniat prædicta Quadrabilitatis conditio; tum per Seriem quæratur Areæ complementum ſ: y d z: quo cognito ſtatim habetur Area quæſita; nam, ut omnibus notum, $zy - ſ: ydz = ſ: zdy$. Et ut ſine confuſione Complementum per Seriem obtineatur; in æquatione data Curvam particularem definiente pro z ſcribatur Y, & pro y ſcribatur Z: Factâque hâc mutatione Ordinatæ in Abſciſſam, & Abſciſſæ in Ordinatam, tractetur æquatio juxta præcepta regulæ ſecundæ; donec illi conveniat Quadrabilitatis conditio, vel eandem ipſi non poſſe convenire pateat.

*Exemplem* 1. Sit $z^3 + y^3 = bzy$. Quia hîc $m = 3$, $n = 3$, $e = 1$, $r = 1$, $a = 1$; ideo $l = 1$, adeoque $l + 1 = 2$. Et proinde (juxta Not. 4.) duo primi Seriei termini dant Aream $= \frac{1}{2} zy - \frac{1}{6} bz^2 y^{-1}$.

*Exemp.* 2. Sit $z^7 + a y^3 = b z y^2$, ubi $m = 7$, $n = 3$, $e = 1$, $r = 2$; qui faciunt $l = 2$; ideo ( juxta Not. 4. ) tres primi Seriei termini dabunt quæſitam

$$\text{AREAM} = \frac{7}{10} z y - \frac{b}{15 a} z^2 - \frac{2 b^2}{15 a^2} z^3 y^{-1}.$$

*Exemp.* 3. Sit $z^3 + k y^5 = h z^{-2} y^{11}$; ubi $m = 1$, $n = 5$, $e = 2$, $r = 11$; at quia hi non faciunt $l$ numerum integrum & affirmativum; ideo ( per Regulam ſecundam ) libero terminum $h z^{-2} y^{11}$ à quantitate $z$; & ſic æquatio fit $z^5 - h y^{11} = - k z^2 y^5$; ubi $a = - h$, $b = - k$; & $m = 5$, $n = 11$, $e = 2$, $r = 5$; qui faciunt $l = 1$: Unde

$$\text{AREA} = \frac{5}{16} z y - \frac{k}{16 h} z^3 y^{-5}$$

*Exemp.* 4. Sit $z^2 - h y^2 = - k z^2 y^2$; ubi $m = 2$, $n = 2$, $e = 2$, $r = 2$; qui non faciunt $l$ numerum integrum & affirmativum; ideo libero terminum $- k z^2 y^2$ à quantitate $z$; & tum $z^0 + k y^2 = h z^{-2} y^2$; ubi $a = k$, $b = h$; & $m = 0$, $n = 2$, $e = - 2$, $r = 2$, qui faciunt $l = 1$, ideo

$$\text{AREA} = \frac{h}{k} z^{-1} y$$

*Exemp.* 5. Sit $z^2 - \frac{4g^2}{h} y^6 = - \frac{g}{h} z^2 y^4$; ubi $m = 2$, $n = 6$, $e = 2$, $r = 4$; qui non faciunt $l$ numerum integrum & affirmativum; idemque contingit liberato ( à quantitate $z$ ) utrolibet ex reliquis: Ideo juxta regulam Tertiam quæro Complementum; quare ( ut jam præmonui ) pono $z = Y$; $y = Z$; unde æquatio data erit

$$Y^2 - \frac{4g^2}{h} Z^6 = - \frac{g}{h} Z^4 Y^2;$$

quæ (juxta Reg. 1.) reducta ad formam generalem erit hujus

modi $Z^6 - \frac{h}{4g^2} Y^2 = \frac{1}{4g} Z^4 Y^2$ ubi $m = 6$, $n = 2$, $e = 4$, $r = 2$; qui non faciunt $l$ numerum integrum & affirmativum; ideo (juxta Reg. 2.) libero terminum ultimum a $Z$; unde $Z^2 - \frac{1}{4g} Y^2 = \frac{h}{4g^2} Z^{-4} Y^2$; ubi $m = 2$, $n = 2$, $e = -4$, $r = 2$; unde $l = 1$; & $a = -\frac{1}{4g}$, $b = -\frac{h}{4g^2}$;
Unde Areæ quæſitæ complementum eſt
$= \frac{1}{2} ZY - \frac{h}{2g} Z^{-3} Y$ ſeu $\frac{1}{2} zy - \frac{h}{2g} zy^{-3}$;
Ergo etiam Area quæſita $\int: zdy = \frac{1}{2} zy + \frac{h}{2g} zy^{-3}$.

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## SECTIO II.

SIT $z^m + ay^n = bz^{2e} y^{2c+n} + fz^e y^{c+n}$ æquatio exprimens Relationem inter Ordinatam $z$ & Abſciſſam $y$. Erit

$$\text{AREA} = Azy + Bz^{e+1} y^{c+1} + Cz^{2e+1} y^{2c+1} + Dz^{3e+1} y^{3c+1} + Ez^{4e+1} y^{4c+1} + Fz^{5e+1} y^{5c+1}, \text{ \&c.}$$

Ubi ( poſitis $2c+n=r$, $c+n=s$) $A=\frac{n}{m+n}$;

$$B=\frac{\overline{\overline{m-e+s}\times A+e-m}}{m\times\overline{c+1}+n\times\overline{e+1}}\times\frac{f}{a}$$

$$C=\frac{\overline{m-2e+r}\times bA+\overline{\overline{m-e}\times\overline{c+1}+r\times\overline{e+1}}\times fB+2eb-mb}{ma\times\overline{2c+1}+na\times\overline{2e+1}}$$

$$D=\frac{\overline{\overline{m-2e}\times\overline{c+1}+r\times\overline{e+1}}\times bB+\overline{\overline{m-e}\times\overline{2c+1}+s\times\overline{2e+1}}\times fC}{ma\times\overline{3c+1}+na\times\overline{3e+1}}$$

$$E=\frac{\overline{\overline{m-2e}\times\overline{2c+1}+r\times\overline{2e+1}}\times bC+\overline{\overline{m-e}\times\overline{3c+1}+s\times\overline{3e+1}}\times fD}{ma\times\overline{4c+1}+na\times\overline{4e+1}}$$

$$F=\frac{\overline{\overline{m-2e}\times\overline{3c+1}+r\times\overline{3e+1}}\times bD+\overline{\overline{m-e}\times\overline{4c+1}+s\times\overline{4e+1}}\times fF}{ma\times\overline{5c+1}+na\times\overline{5e+1}}$$

De hac Serie (cujus progreſſio primo ferè intuitu eſt manifeſta) hæc ſunt notanda. (1.) Quod figuræ (quarum Curvæ prædictâ æquatione definiuntur) ſunt Quadrabiles, quando Numeri exponentiales m, n, e, c; & coefficientes a, b, f habent relationes modo aſſignandas; ſcil: quando $\frac{\overline{2c+m}\times\overline{n-2e}}{-cm-en}$ eſt numerus integer & affirmativus, quem vocemus l. & (cum l eſt major quam 2) quando Coefficientium relatio eſt hæc.

$$\frac{\overline{m-2e}\times\overline{lc-c+1}+r\times\overline{le-e+1}}{\overline{e-m}\times\overline{lc+1}-s\times\overline{le+1}}\times\frac{bU}{f}=$$

$$\frac{\overline{m-2e}\times\overline{lc-2c+1}+r\times\overline{le-2e+1}}{m\times\overline{lc+1}+n\times\overline{le+1}}\times\frac{bP}{a}+$$

$$\frac{\overline{m-e}\times\overline{lc-c+1}+r\times\overline{le-e+1}}{m\times\overline{lc+1}+n\times\overline{le+1}}\times\frac{fU}{a}$$

Ubi U & P denotant Coefficientes Terminorum duorum, qui immediate præcedunt ultimo Areæ quæſitæ Termino; ſcil: U eſt coefficiens termini ad Ultimum propioris, P coefficiens termini ab ultimo remororis: ut ſi $Fz^{5e+1}y^{5c+1}$ eſſet ultimus Areæ quæſitæ terminus, tum U denotaret E, & P denotaret D. (2.) Ultimus ille Areæ quæſitæ terminus ex valore numeri $l$ cognoſcitur; nam hic etiam $l+1$ dat numerum terminorum (ab initio ſumptorum) Seriei, qui Aream quæſitam conſtituunt. (3) Si fuerit $l=1$, tum coefficientium relatio debet eſſe hæc

$$\frac{\overline{2e-m}\times\overline{1-A}+rA}{\overline{e-m}\times\overline{c+1}-s\times\overline{e+1}}\times\frac{b}{f}=\frac{e-m\times\overline{1-A}+sA}{m\times\overline{c+1}+n\times\overline{e+1}}\times\frac{f}{a};$$

Si $l=2$; relatio debet eſſe hæc

$$\frac{m-2e\times\overline{c+1}+r\times\overline{e+1}}{\overline{e-m}\times\overline{2c+1}-s\times\overline{2e+1}}\times\frac{bB}{f}=$$

$$\frac{\overline{2e-m}\times\overline{1-A}+rA}{m\times\overline{2c+1}+n\times\overline{2e+1}}\times\frac{b}{a}+$$

$$\frac{\overline{\overline{m-e}}\times\overline{\overline{c+1}}+s\times\overline{e+1}}{m\times\overline{2c+1}+n\times\overline{2e+1}}\times\frac{fB}{a}$$

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## SECTIO III.

SIT $z^m = ay^n + bz^e y^{c+n} + fz^{2e} y^{2c+n} + gz^{3e} y^{3c+n} + hz^{4e} y^{4c+n} +$ &c. æquatio exprimens relationem inter ordinatam z & abſciſſam y; & conſtans terminis quotcunq; Erit.

$$\text{Area} = Azy + Bz^{e+1} y^{c+1} + Cz^{2e+1} y^{2c+1} + Dz^{3e+1} y^{3c+1} + Ez^{4e+1} y^{4c+1} + \&c.$$

Quod in fallor, eſt Theorema non contemnendum. Calculo per-facili inveniuntur A, B, C, D, E, &c. ut & Quadrabilitatis conditiones, & quot termini ſeriei Aream quæſitam conſtituant. Creſcit quidem numerus harum conditionum pro multitudine terminorum, ex quibus conſtat æquatio relationem inter z & y definiens. Et ſpeciatim ſi illa terminorum multitudo vocetur N; tum N − 2 eſt numerus conditionum Quadrabilitatis; quarum una Exponentium m, n, e, c relationem reſpicit, eſtq; hæc; ut

$$\frac{Nc - 2c + 2e - Ne + m + n}{-cm + en},$$

eſt numerus (quem vocemus l) Integer & affirmativus. Reliquæ vero conditiones coefficientium a, b, f, g, h, &c. reſpiciunt. Ac deniq; l+1 dat numerum terminorum (ab initio ſumptorum) ſeriei, qui Aream quæſitam conſtituunt.

Corol. Ex hac Serie generali deduci poteſt Series, quæ exhibeat Quadraturas Figurarum, quarum Curvæ definiuntur per æquationem conſtantem terminis quibusvis, qui æquationem Sectionis tertiæ generalem conſtituunt. Nam ad hanc obtinendam opus tantum eſt Seriem computare pro æquatione conſtante tot terminis (ab initio ſumptis) æquationis generalis, quot includent Terminos æquatio Curvas definiens. Tum ex valoribus quantitatum A, B, C, D, &c. Eliminentur illæ coefficientes b, f, g, &c. quæ ad æquationem propoſitam non ſpectant ; reliquæ dabunt aream quæſitam Exemplo res datebit.

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## SECTIO IV.

SIT $z^m = ay^n + bz^e y^{c+n} + gz^{3e} y^{3c+n}$

Equationem exprimens relationem inter Z & Y. Jam quia

$$z^m = ay^n + bz^e y^{c+n} + fz^{2e} y^{2c+n} + gz^{3e} y^{3c+n}$$

eſt illa pars pars æquationis quæ (ſumptis terminis in ordine a principio) includit æquationem datam ; quam deinceps (brevitatis cauſa) æquationem completam vocabo ; ideo Figurarum (quarum Curvæ definiuntur per æquationem completam.)

Areæ

$$\text{Areæ} = Azy + Bz^{e+1}y^{c+1} + Cz^{2e+1}y^{2c+1} +$$

$$Dz^{3e+1}y^{3c+1} + Ez^{4e+1}y^{4c+1} + Fz^{5e+1}y^{5c+1} + \&c.$$

& a, b, f, g ingredientur valores quantitatum B, C, D, E, F *&c.* Si ergo in his valoribus ponatur ubiq; $f=0$ (quia $fz^{2e}y^{2c+n}$ æquationem datam non ingreditur) habebis valores quantitatum A, B, C, D, E, *&c.* qui in Serie ſubſtituti dabunt Areas quæſitas. Et Calculo inito in veni.

$$A = \frac{m}{m+n}. \qquad B = \frac{c-\overline{m-c-n\times A}+m-e}{m\times\overline{c+1}+n\times\overline{e+1}}\times\frac{b}{a}$$

$$C = \frac{\overline{c+n}\times\overline{e+1}+\overline{m-e}\times\overline{c+1}}{m\times\overline{2c+1}+n\times\overline{2e+1}}\times\frac{bB}{a}$$

$$D = \frac{\overline{\overline{m-3e}\times\overline{1-A}+\overline{3c+n}\times-A\times g}}{ma\times\overline{3c+1}}$$

$$\frac{+\overline{\overline{m-e}\times\overline{2c+1}+\overline{c+n}\times\overline{2e+1}\times-bC}}{+na\times\overline{3c+1}}$$

B b b b b b b b

E=

$$E=\frac{\overline{\overline{m-3e}\times\overline{c+1}+\overline{3c+n}\times\overline{e+1}}\times -gB+\overline{\overline{m-e}\times\overline{3c+1}+\overline{c+n}\times\overline{3e+1}}\times -bD}{ma\times\overline{4c+1}+na\times\overline{4e+1}}$$

$$F=\frac{\overline{\overline{m-3e}\times\overline{2c+1}+\overline{3c+n}\times\overline{2e+1}}\times -gC+\overline{\overline{m-e}\times\overline{4c+1}+\overline{c+n}\times\overline{4e+1}}\times -bE}{+ma\times\overline{5c+1}+na\times\overline{5e+1}}$$

$$G=\frac{\overline{\overline{m-3e}\times\overline{3c+1}3+\overline{c+n}\times\overline{3e+1}}\times -gD+\overline{\overline{m-e}\times\overline{5c+1}+\overline{c+n}\times\overline{5e+1}}\times -bF}{+ma\times\overline{6c+1}+na\times\overline{6e+1}}$$

Ex his patet progreſſio reliquorum in infinitum. Et ſic habetur Series exhibens Quadraturas omnium Figurarum, quarum Curvæ definiuntur per hanc æquationem quatuor terminorum $z^m = ay^n + bz^e y^{c+n} + gz^{3e} y^{3c+n}$.

Et notandum quod conditiones Quadrabilitatis & numerus terminorum Seriei, Aream quamlibet quæ ſitam conſtituentium, eadem ſunt cum conditionibus Quadrabilitatis, & numero Terminorum, quæ conveniunt Figuris, quarum Curvæ per æquationes completas definiuntur.

Corol. Præter has duas ſeries in § 3 & 4 propter Figuras quatuor terminorum, poſſunt eodem modo infinitæ aliæ ſeries computari pro cæteris caſibus Figurarum, quatuor terminorum. Quod etiam intelligendum eſt De omnibus aliis Figuris, quarum Curvæ per æquationes quotlibet terminorum numero conſtantes definiuntur.

Non jam vacat ipſam Methodum minutiatim deſcribere, per quam ad hujuſmodi Series pervenio; brevem tamen ejus rationem exponere forte non ingratum erit. Aſſumo Seriem ex z pariter ac y compoſitam, ſc: $Azy + Bz^p y^q + Cz^s y^h + Dz^l y^k + \&c. = ſ: zdy$. Cujus ſinguli termini (præter primum) habeant Exponentes incognitos. Tum æquationem inſtituo inter duos valores quantitatis dz, quorum alter ex hac ſerie, alter ex æquatione relationem inter z & y definiente per Methodum Calculi Differentialis directam facile invenitur. Ex terminis hujus æquationis ritè reductæ primo determino

mino exponentes incognitos p, q, g, h, l, k &c. Et de in coefficientes A, B, C, &c. Et ſi plures ſint comparationes, quam quæ determinandis his coefficientibus ſufficiunt, tum ex reliquis deduco Quadrabilitatis conditiones. Si recta ineatur via, Calculus eſt longe facillimus; multaſq; habeo Regulas huc ſpectantes quas alias forſan tradam; ut & uſum hujus Methodi in inveniendis Quadraturis irrationalibus finitis, quando rationales non dantur: res enim omnino in poteſtate eſt.